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LARGE DYNAMIC DISPLACEMENTS AND DEFORMATIONS OF A MASSIVE EXTENSIBLE THREAD IN A MOVING MEDIUM†

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In a Lagrangian system of coordinates, non-linear differential equations are obtained that describe the deformation of a massive extensible thread in a moving medium. Three specific problems with specified numerical parameters are solved by a numerical method developed by the author for solving systems of non-linear Integro-differential equations.

Problems of the deformation of threads and flexible tethered systems arise in the design of different objects in many areas of engineering and technology. Examples are the problem of the shape of a rope when exposed to an air flow [1], the problem of calculating orbital tethered systems [2, 3], and the problem of the deformation of filler fibres in the production of reinforced polymer composites [4]. Below, equations describing the non-linear deformation of a massive thread are obtained and solved for three special cases. The tensile force is examined as a function of the coefficient of relative elongation of the thread, which in turn is connected by a non-linear relation with the components of the displacement vector. The displacements and deformations of the thread are taken to be arbitrarily large.

1. FORMULATION OF THE PROBLEM

It is assumed that, in an orthogonal Cartesian system of coordinates x_i $(i = 1, 2, 3)$, at the initial instant of time $t = 0$ the direction of the originally straight elastic thread with a mass m_0 per unit length is defined by the unit vector $\mathbf{p} = p_i\mathbf{e}_i$ and when $t > t_0$ the motion of the thread occurs in a medium whose velocity field is specified in the form $\mathbf{v}_c = v_{ci}(x_1, x_2, x_3, t)\mathbf{e}_i$. External forces act on the thread, the intensity of which per unit initial length of the thread is equal to

$$\mathbf{F} = F_i(x_1, x_2, x_3, t)\mathbf{e}_i.$$

Taking as the Lagrangian variable the length of an arc of the thread s, it is possible to write expressions for the radius vectors $\mathbf{r}_0$ and $\mathbf{r}$ of an arbitrary point of the thread in the frame of reference x_1 at the initial and actual instants of time, for the displacement vector $\mathbf{w}$, for the basis vectors $\mathbf{g}^0$ and $\mathbf{g}$, and for the relative elongation l [15]

$$\begin{gathered}\mathbf{r}_0 = s(p_i\mathbf{e}_i), \quad \mathbf{w} = w_i\mathbf{e}_i, \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{w} \\ \mathbf{g}^0 = \mathbf{r}_0' = p_i\mathbf{e}_i, \quad \mathbf{g}^0 \cdot \mathbf{g}^0 = 1 \\ \mathbf{g} = \mathbf{r}' = (p_i + w_i')\mathbf{e}_i, \quad l = |\mathbf{g}| - 1\end{gathered} \tag{1.1}$$

A prime denotes differentiation with respect to s.

To obtain the equations of motion at the actual instant of time t, two points of the thread are examined, bounding an infinitesimal elements of the thread ds; the point a, the basis vector of which is $\mathbf{g}$, and the points $\mathbf{a}_1$, for which

$$\mathbf{r}_1 = \mathbf{r} + d\mathbf{r}_0 + d\mathbf{w}, \quad \mathbf{g}_1 = (p_1 + w_i' + dw_i')\mathbf{e}_i \tag{1.2}$$

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The unit vectors of the tangents at points a and a_1 are defined by the expressions

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{g}/|\mathbf{g}|, \quad |\mathbf{g}| = [(p_1+w_1')^2+(p_1+w_2')^2+(p_3+w_3')^2]^{1/2} \\ \boldsymbol{\tau}_1 &= \mathbf{g}_1/|\mathbf{g}_1| = \mathbf{g}/|\mathbf{g}| + \mathbf{g}\{|\mathbf{g}|^{-3}[(p_i+w_i')w_i''] + |\mathbf{g}|^{-1}[w_i''\mathbf{e}_i]\}ds \end{aligned} \tag{1.3}$$

The element ds is acted upon by inertia forces $\mathbf{F}_{in}ds$, external forces $\mathbf{F}ds$, and also forces acting from the medium during longitudinal flow $\mathbf{R}_\tau ds$ and transverse flow $\mathbf{R}_n ds$. The vectors $\mathbf{R}_\tau$ and $\mathbf{R}_n$ are determined using the velocity vector of the motion of the thread relative to the medium $\boldsymbol{\upsilon}_0$, at the actual instant of time

$$\boldsymbol{\upsilon}_0 = \boldsymbol{\upsilon} - \boldsymbol{\upsilon}_c \tag{1.4}$$

where $\boldsymbol{\upsilon} = \dot{w}_i\mathbf{e}_i$ is the velocity of motion of the thread in the system of coordinates x_i (the dot denotes differentiation with respect to time). The vector $\boldsymbol{\upsilon}_0$ can be split into a longitudinal component $\boldsymbol{\upsilon}_{0\tau}$ and a transverse component $\boldsymbol{\upsilon}_{0n}$, and also the normal unit vector $\mathbf{n}$ can be found

$$\boldsymbol{\upsilon}_{0\tau} = (\boldsymbol{\upsilon}_0\cdot\boldsymbol{\tau})\boldsymbol{\tau}, \quad \boldsymbol{\upsilon}_{0n} = \boldsymbol{\upsilon}_0 - \boldsymbol{\upsilon}_{0\tau}, \quad \mathbf{n} = \boldsymbol{\upsilon}_{0n}/|\boldsymbol{\upsilon}_{0n}| \tag{1.5}$$

It is assumed that vectors $\mathbf{R}_\tau$ and $\mathbf{R}_n$ can be represented in the form

$$\mathbf{R}_\tau = -\varphi(|\boldsymbol{\upsilon}_{0\tau}|, \mathrm{Re}, \ldots)\boldsymbol{\tau}, \quad \mathbf{R}_n = -\psi(|\boldsymbol{\upsilon}_{0n}|, \mathrm{Re}, \ldots)\mathbf{n} \tag{1.6}$$

where functions φ and ψ may depend not only on the absolute values of the velocities and on the Reynolds number but also on other parameters of the thread and medium.

At points a and a_1, the thread element ds is acted upon by tensile forces $-T\tau$ and $(T + dT)\tau_1$ respectively, where T is a scalar function of the relative elongation

$$T = \chi(l) \tag{1.7}$$

Equating the principal vector of the forces acting on the thread element to zero, we obtain the equilibrium equation in vector form

$$-T\boldsymbol{\tau} + (T+dT)\boldsymbol{\tau}_1 + (\mathbf{F}_{in}+\mathbf{F}+\mathbf{R}_\tau+\mathbf{R}_n)ds = \mathbf{0} \tag{1.8}$$

which is equivalent to the following three differential equations obtained by scalar multiplication of the left- and right-hand sides of equality (1.8) by $\mathbf{e}_k$

$$\begin{aligned} &(\partial\chi/\partial l)l'|\mathbf{g}|^2(p_k+w_k') + \chi\{|\mathbf{g}|^2w_k'' - (p_k+w_k')[(p_i+w_i')w_i'']\} + \\ &+|\mathbf{g}|^3(-m_0\ddot{w}_k + F_k + R_{\tau k} + R_{nk}) = 0, \quad k = 1, 2, 3 \end{aligned} \tag{1.9}$$

Below it is assumed that the thread moves in the x_1ox_2 plane and at $t = 0$ coincides with the horizontal axis ox. In this special case

$$\begin{aligned} &p_1 = 1, \quad p_2 = p_3 = 0, \quad w_2 = 0, \quad \mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 \\ &x_1 = s + w_1, \quad x_2 = w_2 \end{aligned}$$

Furthermore, the expression for the tensile force is taken in the form

$$T = T_0 + \lambda l \tag{1.10}$$

where T_0 is the initial tension of the thread and λ is a constant parameter. Under these conditions, from system (1.9) it is possible to obtain two equations for calculating the functions $w_1(s, t)$ and $w_2(s, t)$

$$\begin{aligned} &\lambda\alpha(1+w_1') + (T_0+\lambda l)[|\mathbf{g}|w_1'' - \alpha|\mathbf{g}|^{-1}(1+w_1')] + |\mathbf{g}|^2(-m_0\ddot{w}_1 + F_1 - \varphi\tau_1 - \psi n_1) = 0 \\ &\lambda a w_2' + (T_0+\lambda l)[|\mathbf{g}|w_2'' - \alpha|\mathbf{g}|^{-1}w_2'] + |\mathbf{g}|^2(-m_0\ddot{w}_2 + F_2 - \varphi\tau_2 - \psi n_2) = 0 \\ &\alpha = (1+w_1')w_1'' + w_2'w_2'' \end{aligned} \tag{1.11}$$

The system of equations obtained fairly accurately describes the motion of the thread at large displacements and deformations. It is obvious that, if the products of the functions w_1 and w_2 are neglected, the first equation of system (1.11) describes small longitudinal vibrations of a rod, and the second equation describes the transverse vibrations of a string [6].

2. NUMERICAL EXAMPLES

The results of solving the system of equations (1.11), which describe the motion of a thread of length $L_0 = 1$ in certain special cases of boundary and initial conditions are given below. The solution was obtained using a method proposed by the present author, based on the use of interpolation procedures [7] and universal programs for solving two-dimensional boundary-value and initial-boundary-value problems [8].

2.1. The thread is fastened at its ends (at $s = 0$ and $s = 1$). The external forces and the resistance of the medium are ignored. The parameters of the thread are: $T_0 = 1$, $\lambda = 2$, and $m_0 = 10$. At the initial instant of time (at $t = 0$), a velocity is imparted to the particles of the thread in the transverse direction, given by the relation

$$V_0 = \tfrac{1}{2}\sin 2\pi s \tag{2.1}$$

For system of equations (1.11), the initial boundary-value problem was solved with the boundary and initial conditions

$$\begin{aligned} &w_1(0,t) = w_2(0,t) = 0, \quad w_1(1,t) = w_2(1,t) = 0 \\ &w_1(s,0) = w_2(s,0) = 0, \quad \dot{w}_1(s,0) = 0, \quad \dot{w}_2(s,0) = \tfrac{1}{2}\sin 2\pi s \end{aligned} \tag{2.2}$$

A solution was obtained in the form of a sequence of time steps. On the segment $[0, L_0]$, a grid with an odd number of nodes n_s is superimposed. The power of the interpolation polynomial used [8] was taken to be equal to $(n_s + 1)/2$. At each time step, a solution was obtained in a sequence of segments of length H_t, onto which a grid with a number of nodes n_t was superimposed. Consequently, the time step turns out to be equal to $H_t/(n_t - 1)$.

The initial conditions for the solution at the next step were determined from the solution at the preceding step. The iteration process is completed when the discrepancy in Eqs (1.11) reduces to a fixed small value.

A feature of this extremely non-linear problem is that, when $t > 0$, the thread executes free vibrations, and it is therefore possible to track that the law of conservation of energy is satisfied. At any instant of time, the sum of the kinetic and potential energy is equal to

$$\begin{aligned} U &= U_k + U_p = \\ &= \tfrac{1}{2}m_0\int_0^1[(\dot{w}_1(s,t))^2 + (\dot{w}_2(s,t))^2]ds + \int_0^1 l(s)[T_0 + \tfrac{1}{2}\lambda l(s)]ds \end{aligned} \tag{2.3}$$

and this quantity when $t > 0$ should be equal to the value of the kinetic energy of the thread when $t = 0$

$$U_0 = \frac{1}{2}m_0\int_0^1 V_0^2 ds = \frac{5}{8} \tag{2.4}$$

For certain versions of the node grid and three instants of time t, Table 1 presents calculated values, with a time step $h_t = H_t/(n_t - 1) = 0.025$, of the vertical displacement of the middle point of the thread $W_2(1/2, t)$ and the maximum deviation of the total energy from its precise value $\Delta = \max|U - U_0|$, and also the length of the thread in the deformed state when $t = 1.4$, which was calculated from the formula

$$L = \int_0^1 (l+1)ds = \int_0^1 [(1+w_1')^2 + w_2'^2]^{1/2} ds \tag{2.5}$$

Table 1

Grid	$L\ (t = 1.4)$	$w(1/2, t) \times 10^4$			$\Delta \times 10^4$		
		t			t		
		1.4	4.2	9.8	1.4	2.8	9.8
7 × 3	1.4233	4620	−4520	−4240	34	35	58
9 × 3	1.4219	4660	−4580	−4530	11	24	47
13 × 3	1.4239	4684	−4669	–	1	7	–
17 × 3	1.4240	4685	–	–	0.2	–	–

The three values of the time presented in Table 1 were selected such that they approximately corresponded to certain extremum values of the vertical displacement of the middle of the thread. It can be seen that, during deformation, the length of the thread increases by almost half its original value. It is natural that, as time passes, there is an increase in the deviation Δ, but this is small and even at $t = 9.8$ does not exceed 1%. However, if the action of forces of inertia in the horizontal direction is neglected in expression (2.3), the deviation increases to 5%. It must be emphasized that, when $n_s > 9$, and consequently for a high order of approximation of the differential equations by algebraic equations, it proved impossible to obtain a solution of this problem over a long time interval, which seems to be due to the instability of the physical process. At the same time, even when $n_s = 9$ and a power of the interpolating polynomial $n = 4$, the error of approximation is equal to $O(h_s^{n+1}) = O(3 \times 10^{-5})$. In the solution of technical problems, such an error may be acceptable. As regards the power of the interpolating polynomial with respect to time, its minimum value $n = 2$ ($n_t = 3$) is optimal from the viewpoint of the stability of the computing process.

2.2. The right-hand end of the thread with $s = 1$ is securely fixed while the left-hand end moves at constant angular velocity about the right-hand end in a circle of unit radius. The external forces and the resistance of the medium are ignored. The parameters of the thread are: $T_0 = 2$, $\lambda = 4$, and $m_0 = 20$.

The boundary and initial conditions are

$$\begin{aligned} &w_1(0, t) = 1 - \cos(0.2t), \quad w_1(1, t) = 0 \\ &w_1(0, t) = \sin(0.2t), \quad w_1(1, t) = 0 \\ &w_1(s, 0) = w_2(s, 0) = 0, \quad \dot{w}_1(s, 0) = \dot{w}_2(s, 0) = 0 \end{aligned} \tag{2.6}$$

The problem is solved for the following values of the parameters

$$n_s = 5, \quad n_t = 3, \quad h_t = 0.025$$

Figure 1 shows the shape of the thread at various instants of time. At the start of the motion, the thread is distorted by forces of inertia, and then, during rotation about the right-hand stationary point, it executes vibrations, the period of which for the first mode $T_1 \approx 3.35$.

The results of this example confirm that the approach proposed is quite effective when investigating arbitrarily large displacements and deformations of elastic objects, the simplest example of which is a deformable thread.

2.3. The thread is fastened at its ends (at $s = 0$ and $s = 1$). The parameters of the thread are: $T_0 = 1$, $\lambda = 2$ and $m_0 = 20$. The thread is acted upon by an external force

$$\mathbf{F} = 60(1 - x_1/2)\exp(-5t)\mathbf{e}_2$$

The resistance of the medium is taken into account only in the case of transverse flow, and here it is assumed that

$$\boldsymbol{\upsilon}_c = \mathbf{0}, \quad \varphi = 0, \quad \psi = k|\boldsymbol{\upsilon}_{0n}|^2$$

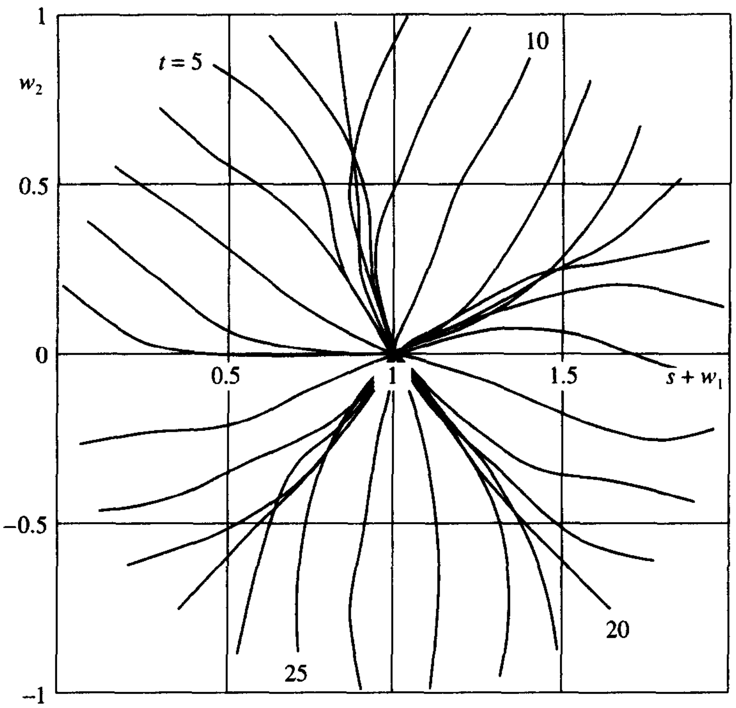

Fig. 1

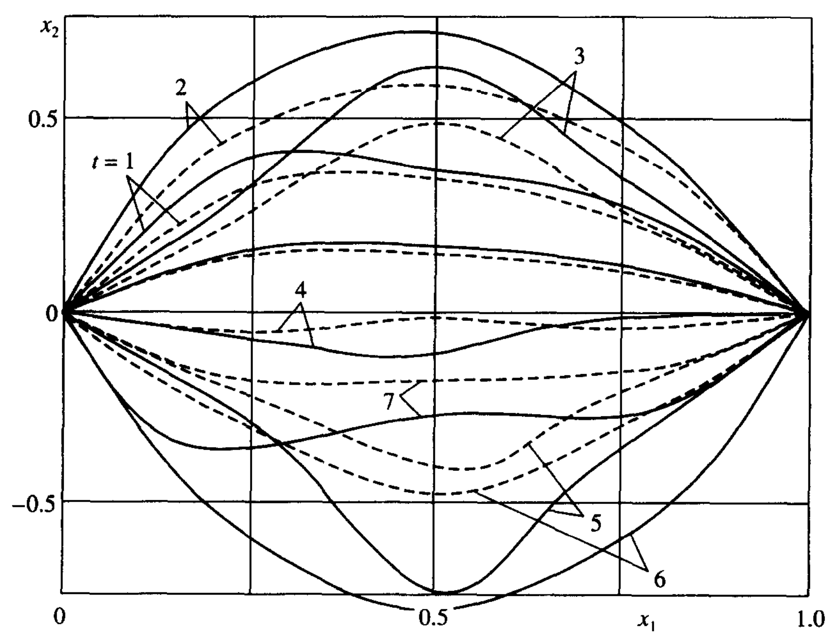

Fig. 2

The expressions for the square of the velocity and the normal unit vector are obtained from relations (1.5)

$$|\mathbf{v}_{0n}|^2 = |\mathbf{v} - (\mathbf{v}_0 \cdot \boldsymbol{\tau})\boldsymbol{\tau}|^2 = (A^2 + B^2)/|\mathbf{g}|^4 \tag{2.7}$$

$$A = \dot{w}_1 w_2'^2 - \dot{w}_2(1 + w_1')w_2', \quad B = \dot{w}_2(1 + w_1')^2 - \dot{w}_1(1 + w_1')w_2'$$

$$\mathbf{n} = (A^2 + B^2)^{-1/2}(A\mathbf{e}_1 + B\mathbf{e}_2)$$

The system of equations (1.11) can now be written in the form

$$\begin{aligned}
&\lambda\alpha(1+w_1')+(T_0+\lambda l)[|\mathbf{g}|w_1''-\alpha|\mathbf{g}|^{-1}(1+w_1')]+[-m_0|\mathbf{g}|^2\ddot{w}_1-kB(A^2+B^2)^{-2}]=0\\
&\lambda\alpha w_2'+(T_0+\lambda l)[|\mathbf{g}|w_1''-\alpha|\mathbf{g}|^{-1}w_2']+[-m_0|\mathbf{g}|^2\ddot{w}_2+\\
&+60(1-(s+w_1)/2)\exp(-5t)-kB(A^2+B^2)^{-2}]=0\\
&\alpha=(1+w_1')w_1''+w_2'w_2''
\end{aligned}\qquad(2.8)$$

The boundary and initial conditions differ from conditions (2.2) only in the fact that, in the case in question, the final condition is also homogeneous: $\dot{w}_2(s, 0) = 0$.

Practically identical results were obtained on grids with $n_x = 5$, $n_t = 3$, $h_t = 0.05$, and $n_x = 7$, $n_t = 3$, $h_t = 0.025$, and they are shown in Fig. 2, where the solid curves represent the positions of the thread at various instants of time t, ignoring the action of the medium ($k = 0$), and the dashed curves represent the positions of the thread when this action is taken into account with $k = 15$. It can be seen that the thread receives large deformations and displacements: at the instants of maximum deviation from the position of static equilibrium; its length is increased by 84%.

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